Complete integrability for Lagrangian dependent on acceleration in a space–time of constant curvature. [1]

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Abstract

The motion equations for a Lagrangian $\mathcal{L}(k_1)$, depending on the curvature k_1 of the particle worldline, embedded in a space–time of constant curvature, are considered and reformulated in terms of the principal curvatures. It is shown that for arbitrary Lagrangian function $\mathcal{L}(k_1)$ the general solution of the motion equations can be obtained by integrals. By analogy with the flat space–time case, the constants of integration are interpreted as the particle mass and its spin. As examples, we completely investigate Lagrangians linear and quadratic in (k_1) and the model of relativistic particle with maximal proper acceleration, in a space–time with constant curvature.

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1 Introduction

Recently, the interest to construct classical models of relativistic particles with spin is again revived (see, for instance, paper [1] and references therein); between the motivations, we remember here the string approach to the theory of elementary particles [2, 3], the construction of the supersymmetric particle models [4], the searching for models of particles with arbitrary fractional spin, the anyons [5], the theory of particles with maximal proper acceleration [6].

A special class of such models are obtained, in particular, by considering Lagrangians depending on higher order derivatives, i.e. on the velocity, the acceleration and other higher derivatives of the particle position vector [7]. The Euler-Lagrange equations in these models written in terms of the position vector turn out to be very complicated already in the free case. An effective method for their investigation has been proposed in a previous paper [8]. This geometrical approach is based on the description of the particle world line by its geometrical invariants, the principal curvatures, rather than by its position vector; the closed set of equations for principal curvatures were derived applying the Hamilton principle, and the general solution obtained in terms of integrals in the case of an arbitrary Lagrangian function $\mathcal{L}(k_1)$ depending on the proper acceleration of the particle, i.e., on the curvature of the world line, k_1.

The investigation of spinning particle interacting with external gauge and gravitational fields is an important task, and, as demonstrated in papers [9, 10], is a nontrivial problem. For instance, the introduction of the external electromagnetic or gravitational fields into the model with Lagrangian $\mathcal{L} = -\alpha\, k_1$ entails the violation of the closure of the constraint algebra.

Therefore it seems to be worthwhile trying to extend the geometrical approach proposed in ref. [8] to the interacting case and, first of all, to the case in which one takes into account the space-time curvature. This is the aim that will be pursued in the present paper. It turns out that new equations of motion, in terms of the principal curvatures of the particle world trajectory, will be derived in space-time of constant curvature. Furthermore, for an arbitrary Lagrangian function $\mathcal{L}(k_1)$ these equations can be exactly integrated; on the analogy of the flat space–time case, one can interpret the integration constants as mass and spin of the particle in the constant curvature space–time.

The layout of the paper is the following. In Sect. 2., after calculating the variation of the proper acceleration in a curved space-time, we derive the equations of motion, generated by the Lagrangian $\mathcal{L}(k_1)$, in terms of the principal curvatures of the world line; the basic tools used here are the Frenet equations for the moving frame along the curve. In Sect. 3. the general solution to the new Euler-Lagrange equations are obtained in terms of integrals. The effectiveness of this method is illustrated by considering some examples: Lagrangians $\mathcal{L}(k_1)$ linear in k_1 and the model of relativistic particle with maximal proper acceleration.[1] In particular, in the last model an important inequality relating the sectional curvature of the space–time, G, and the limiting acceleration of the particle, M_0, is derived very easily: $M_0^2 > G$. In Sect. 4. the results obtained are shortly discussed.

[1] These models have been investigated recently in a *flat space-time* (see references in Sect. 3.).

In Appendix A some mathematical details are presented.

2 Euler-Lagrange equations in terms of the principal curvatures of the world line

We assume that the space-time is an arbitrary D-dimensional Riemannian manifold with the metric tensor $g_{\mu\nu}(x), \quad \mu, \nu = 0, 1, \ldots, D-1$ having the Lorentz signature $(+, -, \ldots, -)$. Reduction to space-time of constant curvature will be made later. In this manifold we shall consider the parametrized curves $x^\mu(s)$ and the generic action

$$S = \int \mathcal{L}(k_1)\, ds \tag{2.1}$$

defined on them. Here $\mathcal{L}$ is an *arbitrary* function of the first curvature of the world line, k_1, i.e., of the proper acceleration of the particle, s is the arclength.[2] It can be shown [11] that any Lorentz and reparametrization invariant action with a Lagrangian function depending on the first and on the second derivatives of the particle position vector can be transformed into the action (2.1).

In order to shorten the formulas, the differentiation with respect to s will be denoted by an overdot and the scalar product generated by the metric tensor $g_{\mu\nu}(x)$ will be denoted by $< \ldots, \ldots >$. We shall also use the notion of *the covariant differentiation along the vector field* $A(x)$: ∇_A. In these notations one has [12, 13]

$$< \dot{x}, \dot{x} > = 1, \tag{2.2}$$

$$k_1^2 = - < \nabla_{\dot{x}}\, \dot{x}, \nabla_{\dot{x}}\, \dot{x} >, \tag{2.3}$$

where $\dot{x}$ is the tangent vector field associated with the particle trajectory $x^\mu(s)$. In the usual notations $\nabla_{\dot{x}}\, \dot{x}$ is nothing else but

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\nu\rho} \frac{dx^\nu}{ds} \frac{dx^\rho}{ds},$$

where $\Gamma^\mu_{\nu\rho}$ are the Christoffel symbols for the metric $g_{\mu\nu}(x)$.

To obtain the equations of motion we, as usually, equate to zero the variation of the action (2.1)

$$\delta S = \delta S_1 + \delta S_2 = 0, \tag{2.4}$$

where

$$\delta S_1 = \int ds\, \delta\mathcal{L}(k_1) = \int ds\, \mathcal{L}'(k_1)\, \delta k_1, \tag{2.5}$$

$$\delta S_2 = \int \mathcal{L}(k_1)\, \delta(ds). \tag{2.6}$$

The prime of the Lagrangian function $\mathcal{L}$ denotes the differentiation with respect to its argument k_1.

[2] More precisely, k_1 is the *geodesic* curvature of the world line in the Riemannian manifold.

Under variation, as usually [13], the position vector of the curve is treated as a function of two variables, $x^\mu(s,\,\xi)$, with the condition $x^\mu(s,\,0)\;=\;x^\mu(s)$. Associated with such a variation is the vector field

$$\xi^\mu(s)\;=\;\frac{\partial x^\mu(s,\,\xi)}{\partial\xi}\bigg|_{\xi\,=\,0} \tag{2.7}$$

defined along the curve $x^\mu(s)$. The variation of an arbitrary function of x, $F(x)$, is given by

$$\delta F\;=\;\frac{\partial F(x(s,\,\xi))}{\partial\xi}\bigg|_{\xi=0}\delta\xi\;=\;\xi^\mu\frac{\partial F}{\partial x^\mu}\,\delta\xi\,, \tag{2.8}$$

or in componentless notation

$$\frac{\partial F}{\partial\xi}\;=\;\xi\circ F\,.$$

In order to calculate the variations, it is convenient to use the Frenet frame associated with the world curve $x^\mu(s)$:

$$\nu_\alpha(s),\;\;\alpha\;=\;0,1,\ldots,D-1, \tag{2.9}$$

$$\nu_0\;=\;\dot{x},\;\;<\nu_\alpha,\,\nu_\beta>=\;\eta_{\alpha\beta}\,,$$

$$\eta_{\alpha\beta}\;=\;\mathrm{diag}\,(1,-1,\ldots,-1),\quad\alpha,\,\beta\;=\;0,1,\ldots,D-1 \tag{2.10}$$

and the Frenet equations describing the motion of this basis along the world line [12]

$$\nabla_{\dot{x}}\,\nu_\alpha\;=\;\omega_\alpha{}^\beta\,\nu_\beta,\quad\omega_{\alpha\beta}\,+\,\omega_{\beta\alpha}\;=\;0\,. \tag{2.11}$$

The rising and lowering the frame indexes α, β, γ, … are made by the constant diagonal tensor $\eta_{\alpha\beta}$ (2.10). Nonzero elements of the matrix ω are determined by the principal curvatures of the world line

$$\omega_{\alpha,\alpha+1}\;=\;-\,\omega_{\alpha+1,\alpha}\;=\;k_{\alpha+1}(s),\quad\alpha\;=\;0,\,1,\,\ldots,\,D-2. \tag{2.12}$$

In the Frenet basis, the variation δx^μ will be defined by the expansion

$$\delta x^\mu(s)\;=\;\xi^\mu(s)\,\delta\xi\;=\;\varepsilon^\alpha(s)\,\nu^\mu_\alpha(s)\,, \tag{2.13}$$

$$\mu\;=\;0,\,1,\,\ldots,\,D\,-\,1,\quad\alpha\;=\;0,\,1,\,\ldots,\,D-1,$$

where $\varepsilon^\alpha(s)$ are arbitrary functions.

For the variation δS_2 we have

$$\delta S_2\;=\;\int\mathcal{L}(k_1)\,\dot{x}^\nu\,g_{\mu\nu}\,d(\delta x^\mu)\;+\;\frac{1}{2}\int ds\,\mathcal{L}\,\dot{x}^\mu\,\dot{x}^\nu\,\frac{\partial g_{\mu\nu}}{\partial x^\lambda}\,\delta x^\lambda\,. \tag{2.14}$$

Here we have used the commutativity of the symbols δ and d. Partially integrating in the first term and dropping the terms outside the integral, as it is always done when deriving the equations of motion from the Hamiltonian principle, we arrive at the formula

$$\delta S_2\;=\;-\int\mathcal{L}'\,\dot{k}_1<\dot{x},\,\delta x>ds\;-\;\int\mathcal{L}<\nabla_{\dot{x}}\dot{x},\,\delta x>ds\,. \tag{2.15}$$

Taking into account the Frenet equations (2.11) and the expansion (2.13), the variation δS_2 acquires the form

$$\delta S_2 \,=\, - \int \mathcal{L}'(k_1)\, \dot{k}_1\, \varepsilon^0(s)\, ds \,-\, \int \mathcal{L}(k_1)\, k_1(s)\, \varepsilon^1(s)\, ds\,. \tag{2.16}$$

Let us now turn to the calculation of the variation δS_1 defined by (2.5). At this end we firstly calculate the variation of the first curvature $k_1(s)$ in terms of δx^μ. For two arbitrary vector fields $A(x)$ and $B(x)$, the following equality holds:

$$\frac{\partial}{\partial x^\mu} < A,\, B > = < \nabla_\mu A,\, B > \,+\, < A,\, \nabla_\mu B > \tag{2.17}$$

Making use of (2.3), (2.8) and (2.17), we obtain the following expression for the variation of the curvature $k_1(s)$

$$\delta k_1^2 \,=\, 2\, k_1\, \delta k_1 \,=\, -\, 2 < \nabla_{\dot{x}}\, \dot{x},\, \nabla_\xi \nabla_{\dot{x}}\, \dot{x} > \delta\xi\,. \tag{2.18}$$

Commutator of two covariant derivatives can be expressed in terms of the Riemann curvature tensor. For three vector fields A, B, and C we have the first structure equation [13]

$$\nabla_A\, \nabla_B\, C \,-\, \nabla_B\, \nabla_A\, C \,-\, \nabla_{[A,\,B]}\, C \,=\, R(A,\, B)\, C\,, \tag{2.19}$$

where $[A,\, B]$ is the commutator of the vector fields A and B

$$[A,\, B] \,=\, \left(A^\nu \frac{\partial B^\mu}{\partial x^\nu} \,-\, B^\nu \frac{\partial A^\mu}{\partial x^\nu} \right) \frac{\partial}{\partial x^\mu}\,. \tag{2.20}$$

This enables us to substitute $\nabla_\xi\, \nabla_{\dot{x}}\, \dot{x}$ in (2.18) by the following expression

$$\nabla_\xi\, \nabla_{\dot{x}}\, \dot{x} \,=\, \nabla_{\dot{x}}\, \nabla_\xi\, \dot{x} \,+\, \nabla_{[\xi,\, \dot{x}]}\, \dot{x} \,+\, R(\xi,\, \dot{x})\, \dot{x}\,. \tag{2.21}$$

The torsion of the space-time is assumed to be zero. Therefore the second structure equation reads [13]

$$\nabla_\xi \dot{x} \,-\, \nabla_{\dot{x}}\, \xi \,-\, [\xi,\, \dot{x}] \,=\, 0\,. \tag{2.22}$$

In Appendix A it is shown that

$$[\xi,\, \dot{x}] \,=\, -\, < \dot{x},\, \nabla_{\dot{x}}\, \xi > \dot{x}\,. \tag{2.23}$$

Hence, eq. (2.21) acquires the form[3]

$$\begin{aligned} \nabla_\xi\, \nabla_{\dot{x}}\, \dot{x} \quad &= \quad \nabla_{\dot{x}}\, \nabla_{\dot{x}}\, \xi \,-\, 2 < \dot{x},\, \nabla_{\dot{x}}\, \xi > \nabla_{\dot{x}}\, \dot{x} \,- \\ &\quad -\, \frac{d}{ds}\, (< \dot{x},\, \nabla_{\dot{x}}\, \xi >)\, \dot{x} \,+\, R(\xi,\, \dot{x})\, \dot{x}\,. \end{aligned} \tag{2.24}$$

[3]Operator $\nabla_{\dot{x}}$ acting on the scalar function reduces to the usual differentiation d/ds.

Substituting (2.24) into (2.18) and taking into account that $< \nabla_{\dot x}\, \dot x,\, \dot x >= 0$, we can write

$$
\begin{aligned}
k_1\, \delta k_1 \;&=\; -\,\delta\xi\, \{< \nabla_{\dot x}\, \dot x,\, \nabla_{\dot x}\, \nabla_{\dot x}\, \xi > + \\
&\quad + 2\, k_1^2 < \dot x,\, \nabla_{\dot x}\, \xi > + < \nabla_{\dot x}\, \dot x,\, R(\xi,\, \dot x)\, \dot x > \Big\}\,. \qquad (2.25)
\end{aligned}
$$

It should be noted here that $\delta\xi$ in eq. (2.25) is a variation of the independent variable which enters as an argument of the function $x^\mu(s,\, \xi)$. Hence, $\delta\xi$ obviously commutes with the differentiation operator $\nabla_{\dot x}$. Now we expand $\xi^\mu\, \delta\xi$ in (2.25) according to (2.13) and use the Frenet equations (2.11). As a result, the variation $\delta k_1(s)$ can be represented as

$$
\begin{aligned}
\delta k_1 \;&=\; \varepsilon^0\, \dot k_1 \,-\, \ddot\varepsilon^1 \,+\, \varepsilon^1\, \left(k_1^2 \,+\, k_2^2\right) \,-\, 2\, \dot\varepsilon^2\, k_2 \,-\, \varepsilon^2\, \dot k_2 \,- \\
&\quad -\, \varepsilon^3\, k_2\, k_3 \,+\, \sum_{\alpha=0}^{D-1} \varepsilon^\alpha(s) < \nu_1,\, R(\nu_\alpha,\, \nu_0)\nu_0 > \,. \qquad (2.26)
\end{aligned}
$$

In the component notations the last term in (2.26) is written as

$$
\sum_{\alpha=0}^{D-1} \varepsilon^\alpha(s)\, < \nu_1,\, R(\nu_\alpha,\, \nu_0)\, \nu_0 >= \varepsilon^\alpha(s)\, R_{\mu\nu\rho}{}^{\sigma} \nu_\alpha^\mu\, \nu_0^\nu\, \nu_0^\rho\, \nu_{1\sigma}\,.
$$

Thus for an arbitrary Riemann curvature tensor $R_{\mu\nu\rho}{}^{\sigma}$, the variation δk_1 is expressed not only in terms of the principal curvatures of the world line, $k_j(s),\;\; j \,=\, 1,\, 2,\, 3$ but it also depends on the normals $\nu_\alpha(s),\;\; \alpha \,=\, 0,\, 1,\, \ldots,\, D-1$ to the curve. As a consequence, for an arbitrary curvature tensor $R_{\mu\nu\rho}{}^{\sigma}$, we cannot derive a closed set of equations of motion containing only $k_j(s)$; however, for special Riemannian manifolds, dependence on the Frenet frame in (2.26) may disappear. For example, the space-time of constant sectional curvature G has the Riemann tensor defined by [13]

$$
R_{\mu\nu\lambda\rho} \,=\, G\, (g_{\mu\rho}\, g_{\nu\lambda} \,-\, g_{\mu\lambda}\, g_{\nu\rho})\,.
$$

Scalar curvature $R \,=\, g^{\mu\rho}\, g^{\nu\lambda}\, R_{\mu\nu\lambda\rho}$ is related to the sectional curvature G in the following way $R \,=\, D\, (D \,-\, 1)\, G$, where D is the dimension of the space-time. Now we have

$$
\sum_{\alpha=0}^{D-1} \varepsilon^\alpha(s) < \nu_1,\, R(\nu_\alpha,\, \nu_0)\, \nu_0 >= -\, G\, \varepsilon^1(s)\,. \qquad (2.27)
$$

Taking into account (2.26) and (2.27), the variation δS_1 can be represented in the form

$$
\begin{aligned}
\delta S_1 \;&=\; \int ds\, \Big\{\mathcal{L}'(k_1)\, \dot k_1\, \varepsilon^0(s) + \\
&+ \left[(k_1^2 \,+\, k_2^2)\, \mathcal{L}'(k_1) \,-\, \frac{d^2}{ds^2}\, (\mathcal{L}'(k_1)) \,-\, \mathcal{L}'(k_1)\, G\right] \varepsilon^1(s) \qquad (2.28) \\
&+ \left[2\, \frac{d}{ds}\, (\mathcal{L}'(k_1)\, k_2) \,-\, \dot k_2\, \mathcal{L}'(k_1)\right] \varepsilon^2(s) \,-\, \mathcal{L}'(k_1)\, k_2\, k_3\, \varepsilon^3(s) \Big\}\,.
\end{aligned}
$$

Summing eqs. (2.14) and (2.28), we obtain the set of three equations for principal curvatures k_1, k_2, and k_3 (terms containing $\varepsilon^0(s)$ in δS_1 and in δS_2 are canceled):

$$\begin{aligned} \frac{d^2}{ds^2}\left(\mathcal{L}'(k_1)\right) &= \left(k_1^2 + k_2^2 - G\right)\mathcal{L}'(k_1) - k_1\,\mathcal{L}(k_1)\,, & (2.29)\\ 2\,\frac{d}{ds}\left(\mathcal{L}'(k_1)\,k_2\right) &= \dot{k}_2\,\mathcal{L}'(k_1)\,, & (2.30)\\ \mathcal{L}'(k_1)\,k_2\,k_3 &= 0\,. & (2.31) \end{aligned}$$

It is remarkable that the constant sectional curvature of the space-time, G, enters only in eq. (2.29). Equations (2.30) and (2.31) remain the same as in the flat space-time [8].

In order to satisfy eq. (2.31) we put $k_3(s) = 0$. Then all the higher curvatures will vanish too [14]. Thus, for arbitrary D we have

$$k_n(s) = 0, \quad n = 3, 4, \ldots, D-1\,. \tag{2.32}$$

Equation (2.30) can be integrated

$$\left(\mathcal{L}'(k_1)\right)^2 k_2 = C\,, \tag{2.33}$$

where C is an integration constant. Taking into account (2.33), we remain with one nonlinear equation of the second order for the curvature $k_1(s)$

$$\frac{d^2}{ds^2}\left(\mathcal{L}'(k_1)\right) = \left(k_1^2 + \frac{C^2}{\left(\mathcal{L}'(k_1)\right)^4} - G\right)\mathcal{L}'(k_1) - k_1\,\mathcal{L}(k_1)\,. \tag{2.34}$$

After integrating this last equation, we can reconstruct all the principal curvatures of the world line $k_i(s)$, $i = 1, 2, \ldots, D-1$. From the classical differential geometry [12], it is well known that the principal curvatures of a world line in a background of constant sectional curvature determine this curve up to its transformations as a whole, which are given by the symmetry group of the enveloping space (in the case under consideration it is the $SO(1, D-1)$ group). Obviously, this specification of the world line should take into account all the essential physical properties of the model in question.

3 Exact integrability of the Euler-Lagrange equations for principal curvatures. Examples

The first integral for equation (2.34) can be found directly. As a result the problem of constructing the general solution to this equation is reduced to quadratures. In Ref. [8] in the case of a flat space–time, this integral has been derived by investigating the equations of motion, generated by action (2.1), in terms of the position vector x^μ of the world line. To accomplish similar calculations in the case under consideration is a rather complicated task. Nevertheless knowing the integral for eq. (2.34) at $G = 0$ one can construct such

an integral at $G \neq 0$ too. Really, by a direct differentiation one can convinced oneself that the expression

$$M^2 = \mathcal{L}^2 - \left(\frac{d}{ds}\mathcal{L}'\right)^2 - 2\,\mathcal{L}\,\mathcal{L}'\,k_1 + \left(\mathcal{L}'\right)^2\,k_1^2 - \frac{C^2}{(\mathcal{L}')^2} - G\,\left(\mathcal{L}'\right)^2\,, \tag{3.1}$$

M^2 being an integration constant, is the first integral of the eq. (2.34) if $\dot{k}_1\mathcal{L}'' \neq 0$. For Lagrangians linear in k_1, eqs. (3.1) and (2.34) should be treated as independent ones, because in this case the differentiation of (3.1) gives identically zero. From (3.1) we obtain

$$\frac{dk_1}{ds} = \pm\sqrt{f(k_1)}\,, \tag{3.2}$$

where

$$f(k_1) = \frac{1}{(\mathcal{L}'')^2}\left\{\mathcal{L}^2 - 2\,\mathcal{L}\,\mathcal{L}'\,k_1 + (\mathcal{L}')^2\,(k_1^2 - G) - \frac{C^2}{(\mathcal{L}')^2} - M^2\right\}. \tag{3.3}$$

Integration of eq. (3.2) yields

$$\int\limits_{k_1(s_0)}^{k_1(s)} \frac{dx}{\sqrt{f(x)}} = \pm\,(s - s_0)\,. \tag{3.4}$$

Thus, making use of (3.4), one can obtain the first curvature $k_1(s)$ of the world trajectory for any given Lagrangian function $\mathcal{L}(k_1)$. Then eq. (2.33) enables one to find the torsion $k_2(s)$ of the world line, all remaining principal curvatures being equal to zero identically. Hence, the problem of solving the equations of motion reduces to doing the integral (3.4).

As it was shown in Ref. [8] for a flat space-time, the constant of integration M^2 turns out to be the mass squared of the particle[4] and the second integration constant, C, determines the partcle spin S

$$S^2 = \frac{C^2}{|M^2|} = \frac{k_2^2\,(\mathcal{L}'(k_1))^4}{|M^2|}\,. \tag{3.5}$$

Here the absolute value of M^2 is taken, because in general these models may also have tachyonic solutions.

For a curved space-time and in particular for space-time with constant curvature there is no unique prescription for defining the particle mass and spin. Therefore on the analogy of the flat space-time case, one can treat eqs. (3.1) and (3.5) as definitions of the particle

[4]In particle models with action containing higher derivatives, the particle mass and spin are, at classical level, simply the *integrals of motion*, whose values are determined by the initial conditions for the corresponding Euler-Lagrange equations. Therefore these integrals may, in principle, acquire *arbitrary values*. Only upon quantization, in some models of this kind one can obtain *discrete values* for M^2 and S [7, 15].

mass and spin in the case of particles with action (2.1), moving in constant curvature space–time. Applying these formulas to the usual scalar particle with action

$$S_0 \,=\, -\, m \int ds$$

one obtains $M^2 \,=\, m^2$ and $S \,=\, 0$, i.e., mass and spin of the particle remain the same as in the flat space-time.

It is worthwhile to note that our equations (2.29)–(2.31), as well as (3.1) and (3.5), solve the variational problem (2.1) (2.4) in a class of *regular curves* $x^\mu(s)$, i.e., in the class in which the standard Frenet frame (2.9), (2.10) can be associated with each point of the world line.

Let us now analyse some examples. We begin with the Plyushchay model of the massless spinning particle defined by the Lagrangian [15, 16]

$$\mathcal{L} \,=\, -\, \alpha\, k_1(s)\,. \tag{3.6}$$

In the Minkowski space-time the principal curvatures are:

$$\begin{gathered} k_1(s) \text{ is an arbitrary function of } s \text{ and} \\ k_2(s) \,=\, k_3(s) \,=\, \ldots = \, k_{D-1}(s) \,=\, 0\,. \end{gathered} \tag{3.7}$$

The solutions obtained in [15], in form of helical curves, have just zero torsion and constant curvature. The last condition can be treated as a consequence of the gauge fixing. The essential point is that the conditions $k_2(s) \,=\, 0$ and $k_1(s) \,=$constant imply superlight velocities. Really, from the Frenet equations (2.11), which are at the same time the definitions of the principal curvatures k_j, we obtain when $k_2 \,=\, 0$

$$\dddot{x}^2 \,=\, -\, \dot{k}_1^2 \,+\, k_1^4\, \dot{x}^2\,. \tag{3.8}$$

From physical considerations, the vectors $\ddot{x}$ and $\dddot{x}$ should be treated naturally as space-like vectors (this follows immediately in the gauge in which the evolution parameter is x^0). In view of this, one deduces from (3.8) that the vector $\dot{x}^\mu$ must be space–like[5] when $\dot{k}_1 \,=\, 0$.

In the case of constant curvature space-time, the solutions to eqs. (2.29)–(2.31) for the Lagrangian (3.7) are

$$\begin{gathered} k_1(s) \text{ remains arbitrary and} \\ k_2^2 \,=\, G, \;\; k_3(s) \,=\, k_4(s) \,= \ldots =\, k_{D-1}(s) \,=\, 0\,. \end{gathered} \tag{3.9}$$

Hence, the theory will be consistent only for space-time with constant curvature G. Probably, this point is related to the unclosure of the constraints algebra which has been recovered in Ref. [10] by considering the Hamiltonian formalism for this model in a curved space-time.

[5]In Ref. [8], as well as in the present paper, it has been assumed that the world lines must be time-like ($\nu_0^2 \,=\, 1$). However, taking into account the remark made before eqs. (2.29)–(2.31), it is obvious that these equations remain the same in the case of space-like curves too.

Using eqs. (3.1) and (3.5) proposed by us for the particle mass and spin in space-time of constant sectional curvature G, we obtain for the Lagrangian (3.7)

$$M^2 \;=\; -2\,\alpha^2 G, \tag{3.10}$$

$$S^2 \;=\; \alpha^2/2\,. \tag{3.11}$$

As it was noted above, the consistency of the model requires $G \,\geq\, 0$. Therefore, from eq. (3.10) it follows that all the solutions in this model are tachyonic with $M^2 \,<\, 0$. From (3.10) and (3.11) we obtain the relation between M^2 and S^2

$$M^2 \;=\; -\,4\,G\,S^2\,. \tag{3.12}$$

In quantum case the parameter α, according to (3.11), becomes discrete

$$\alpha^2 = 2\,S\,(S + D - 1), \qquad S = 0, 1, \ldots\,. \tag{3.13}$$

Let us now consider the model defined by the Lagrangian

$$\mathcal{L}(k_1) \;=\; -\,m \;-\; \alpha\,k_1(s). \tag{3.14}$$

In a flat space this model has been examined in papers [7, 17]. From the equations of motion (2.29)–(2.31) it follows that in the space-time of constant curvature the principal curvatures k_1 and k_2 are constants obeying the relation

$$\alpha\,k_2^2 \;=\; \alpha\,G \;+\; m\,k_1\;. \tag{3.15}$$

All the other curvatures are equal to zero. Equations (3.1) and (3.5) give for the mass and the spin of the particle

$$M^2 \;=\; \frac{m^2 - \alpha^2\,G}{1 \,+\, \alpha^{-2} S^2}\;, \tag{3.16}$$

$$\tag{3.17}$$

$$S^2 \;=\; \frac{\alpha^4\,k_2^2}{M^2}\,. \tag{3.18}$$

Thus, the curvatures k_1 and k_2 are ultimately expressed in terms of M^2 and S^2. Equation (3.16) is the spin-mass relation. As compared with the flat space-time there appears an additional term $-\alpha^2\,G$ in eq. (3.16).

Let us consider the rigid relativistic particle with Lagrangian

$$\mathcal{L} \;=\; -\,m - \alpha\,k_1^2\,. \tag{3.19}$$

In flat space-time this model has been investigated in papers [11, 17, 21]. In space-time of constant curvature the equations of motion for $k_j(s)$ read

$$\begin{aligned} &2\,\ddot{k}_1 \,+\, k_1^3 \,-\, 2\,k_1\,k_2^2 \,+\, 2\,G\,k_1 \,+\, \alpha^{-1} k_1\,m \;=\; 0\;, \\ &16\,\alpha^4\,k_1^4\,k_2^2 \;=\; S^2\,M^2\;, \\ &k_3 \;=\; k_4 \;=\; \ldots \;=\; k_{D-1} = 0\;. \end{aligned} \tag{3.20}$$

Applying general formula (3.4), the solution to (3.19) can be expressed in terms of elliptic integrals, in complete analogy with previous investigations of this model in flat space-time [14, 21]. Therefore we do not present here the corresponding formulas.

And finally we consider the model of relativistic particle with maximal proper acceleration [6]

$$\mathcal{L}(k_1) = -\mu_0\sqrt{M_0^2 - k_1^2}\ , \tag{3.21}$$

where $\mu_0 = m/M_0$ and M_0 is the upper value of the proper acceleration of the particle.[6] Obviously, for physical applications it is interesting to investigate the behaviour of $k_1(s)$ near the boundary M_0. In this region equations (3.2)–(3.4) give

$$\int\limits_{k_1(\bar{s}_0)}^{k_1(\bar{s})} \frac{dk}{\sqrt{1-k^2}} \;=\; \pm\sqrt{1\,-\,g}\,(\bar{s}\,-\,\bar{s}_0)\ \ , \tag{3.22}$$

where $k(\bar{s})$ is a dimensionless acceleration, $k^2 \,=\, k_1^2/M_0$, $\bar{s} \,=\, s\,M_0$ is the dimensionless arclength, g stands for the ratio of the sectional curvature of the background space–time to the squared maximal acceleration M_0^2, $g \,=\, G/M_0^2$. From (3.21) it follows that the model under consideration is consistent only when[7]

$$M_0^2 \,>\, G\ . \tag{3.23}$$

Integration in (3.21) gives

$$k^2(\bar{s}) \;=\; \tanh^2\left[\sqrt{1\,-\,g}(\bar{s}\,-\,\bar{s}_0)\right],\quad \bar{s}\,\rightarrow\,\pm\infty\ . \tag{3.24}$$

Hence, the restriction

$$k^2(\bar{s})\,<\,1$$

always holds.

4 Conclusion

The geometrical approach for investigating the models with action (2.1) in space-time of constant curvature, proposed here, enables us, without complicated calculations, to reveal the basic important particularities in this dynamics. As well known, the reparametrization invariant actions, like (2.1), give rise to constrained dynamics [7], both in Lagrangian and Hamiltonian settings. The analysis of the constraints and the choice of an appropriate gauge fixing condition turn out to be a rather complicated task. In this regard, it is remarkable that our approach allows us to avoid this problem. Besides, in the framework of the geometrical treatment, there appears a possibility for introducing in a consistent

[6] In ref. [23] this Lagrangian has been investigated in another context.

[7] The sectional curvature of the space-time, G, is supposed to be positive. The space-time with $G < 0$ has rather unusual properties; for example, the time-like closed geodesic curves may exist there [22].

way the definition of particle mass and spin in space-time of constant curvature as special integrals of motion. Certainly, it is interesting to elucidate the relationship between these definitions and other approaches to this problem.

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Appendix A

By making use of the definition of the commutator (or the Lie bracket) of two vector fields (2.20), we can write

$$[\xi,\,\dot{x}] \,=\, \left(\xi^\nu \,\frac{\partial \dot{x}^\mu}{\partial x^\nu} \,-\, \dot{x}^\nu\,\frac{\partial \xi^\mu}{\partial x^\nu}\right)\,\frac{\partial}{\partial x^\mu} \,=$$

$$=\, \frac{\partial}{\partial \xi}\,\frac{d}{ds} \,-\, \frac{d}{ds}\,\frac{\partial}{\partial \xi} \,=\, -\,\frac{1}{ds}\,\left(\frac{\partial}{\partial \xi}\,ds\right)\,\frac{d}{ds}\,. \tag{A.1}$$

Further we have

$$\frac{\partial}{\partial \xi}\,ds \,=\, \frac{\partial}{\partial \xi}\sqrt{dx^\mu\,dx^\nu\,g_{\mu\nu}(x)} \,=\, \left(\dot{x}^\mu\,\dot{\xi}^\nu\,g_{\mu\nu} \,+\, \frac{1}{2}\,\frac{\partial g_{\mu\nu}}{\partial x^\rho}\,\xi^\rho\,\dot{x}^\mu\,\dot{x}^\nu\right)\,ds\,. \tag{A.2}$$

In eq. (A.2) $\partial g_{\mu\nu}/\partial x^\rho$ can be substituted by

$$\frac{\partial g_{\mu\nu}}{\partial x^\rho} \,+\, \frac{\partial g_{\rho\mu}}{\partial x^\nu} \,-\, \frac{\partial g_{\nu\rho}}{\partial x^\mu}$$

because the additional terms are cancelled. As a result, eq. (A.2) transforms to

$$\frac{\partial}{\partial \xi}\,ds \,=\, \dot{x}^\lambda\,g_{\lambda\mu}\,\left(\dot{\xi}^\mu \,+\, \Gamma^\mu_{\nu\rho}\,\xi^\rho\,\dot{x}^\nu\right)\,ds \,=<\dot{x},\,\nabla_{\dot{x}}\,\xi>ds\,. \tag{A.3}$$

Finally we obtain

$$[\xi,\,\dot{x}] \,=\, -\,<\dot{x},\,\nabla_{\dot{x}}\,\xi>\,\frac{d}{ds} \,=\, -<\dot{x},\,\nabla_{\dot{x}}\,\xi>\dot{x}\,, \tag{A.4}$$

where $\dot{x}$ should be treated as a vector field.

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